LECTURE 2

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Linear Equations and Inequalities

(Introduction to affine geometry)

Notations:

$$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}, \quad \mathbb{R}_> = \{x \in \mathbb{R} \mid x > 0\}$$
$$\mathbb{R}_- = \{x \in \mathbb{R} \mid x \leq 0\}, \quad \mathbb{R}_< = \{x \in \mathbb{R} \mid x < 0\}$$

Definition 1. Let $a \in \mathbb{R}^p \backslash \{0\}$. The set

$$H_a = \{x \in \mathbb{R}^p \mid \langle x, a \rangle = 0\}$$

is called the **hyperplane through origin with normal vector** a, where $\langle \cdot, \cdot \rangle$ is the **standard scalar product** in $\mathbb{R}^p$, defined as:

$$\langle (x_1, ...x_p), (a_1, ..., a_p) \rangle := \sum_{i=1}^{p} x_i a_i.$$

Further H_a defines two **closed halfspaces**:

$H_a^+ = \{x \in \mathbb{R}^d \mid \langle x, a \rangle \geq 0\}$
$H_a^- = \{x \in \mathbb{R}^d \mid \langle x, a \rangle \leq 0\} = H_{-a}^+$

and two **open halfspaces**:

$H_a^> = \{x \in \mathbb{R}^d \mid \langle x, a \rangle > 0\}$
$H_a^< = \{x \in \mathbb{R}^d \mid \langle x, a \rangle < 0\} = H_{-a}^>$

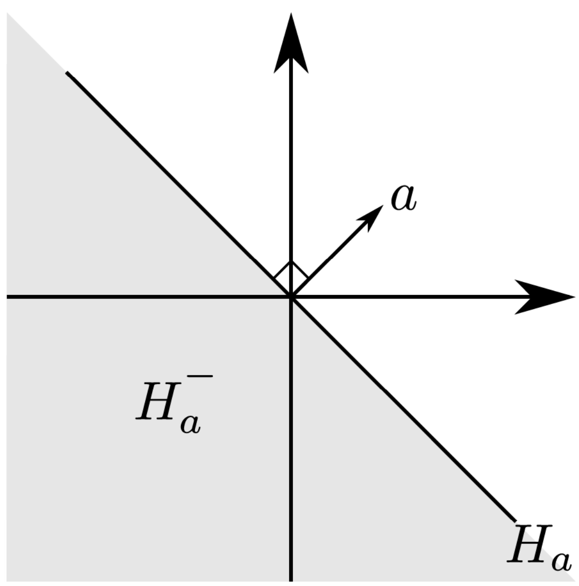

FIGURE 1. Hyperplane through origin with normal vector a

Definition 2. A function $\alpha : \mathbb{R}^d \to \mathbb{R}$ with

$$\alpha(x) = \langle x, a \rangle + a_0, a_0 \in \mathbb{R}, a \in \mathbb{R}^d \backslash \{0\}$$

is called an **affine form**.

For an affine form we set the **hyperplane**

$$H_\alpha = \{x \in \mathbb{R}^d \mid \alpha(x) = 0\}$$

and we define as above $H_\alpha^+, H_\alpha^-, H_\alpha^>$ and $H_\alpha^<$.

Example 3. Assume $a_0 > 0$ and let $a \in \mathbb{R}^d$. The hyperplanes corresponding to the affine forms $\langle x, a \rangle + a_0$ and $\langle x, a \rangle - a_0$ are illustrated in Figure 2.

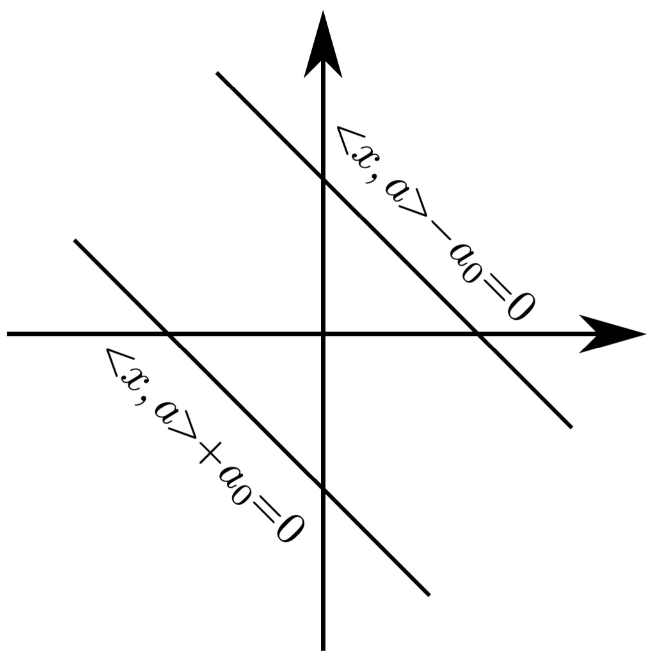

FIGURE 2. Hyperplanes induced by affine forms

Definition 4. A subset $S \subset \mathbb{R}^p$ is called an **affine subspace** of $\mathbb{R}^p$ if it is closed under all **affine linear combinations**, i.e. for all $s_1, ..., s_n \in \mathbb{R}$ such that $\sum_{i=1}^n s_i = 1$ and all $x_1, ..., x_n \in S$ we have that

$$\sum_{i=1}^n s_i x_i \in S.$$

Proposition 5. *A hyperplane is an affine subspace.*

Proof. Let $x_1, ..., x_n \in H_\alpha$. We have that $\alpha(x_i) = 0, \ \forall i = \overline{1, n}$. This is equivalent to

$$< x_i, a > +a_0 = 0, \ \forall i = \overline{1, n}.$$

Let $s_i \in \mathbb{R}$ such that $\sum_{i=1}^n s_i = 1$. Then

$$0 = \sum_{i=1}^n s_i(\langle x_i, a\rangle + a_0) \iff$$
$$0 = \sum_{i=1}^n s_i\langle x_i, a\rangle + \sum_{i=1}^n s_i a_0 \iff$$
$$0 = \sum_{i=1}^n s_i\langle x_i, a\rangle + a_0 \sum_{i=1}^n s_i \iff$$
$$0 = \sum_{i=1}^n s_i\langle x_i, a\rangle + a_0 \underbrace{\sum_{i=1}^n s_i}_{=1} \iff$$
$$0 = \langle \sum_{i=1}^n s_i x_i, a\rangle + a_0 \implies \sum_{i=1}^n s_i x_i \in H_\alpha$$

□

Notation: Let $X \subset \mathbb{R}^d$ be a subset. Then we use **aff**(X) to denote the **affine hull** of X, i.e. the smallest affine subspace of $\mathbb{R}^p$ containing X.

Definition 6. Let $x_i \in \mathbb{R}^p$, $i = \overline{0, n}$ be a set of points. Assume that for any

$$x = \sum_{i=0}^n a_i x_i \in \mathbf{aff}(x_0, x_1, ..., x_n)$$

the elements a_i are *uniquely determined* such that $\sum_{i=0}^i a_i = 1, \forall i = \overline{0, n}$. Then we say that $x_0, ..., x_n$ are **affinely independent**.

If $x_0, ..., x_n$ are affinely independent we say that:

$$\dim \mathbf{aff}(x_0, ..., x_n) = n$$

Remark 7. A hyperplane in $\mathbb{R}^p$ is the affine hull of a p affinely independent point $\implies$ the (affine) dimension of a hyperplane is $p - 1$.

Example 8. Let $x^1 = (1, 0)$ and $x^2 = (0, 1)$. Compute aff$(\{x^1, x^2\})$.

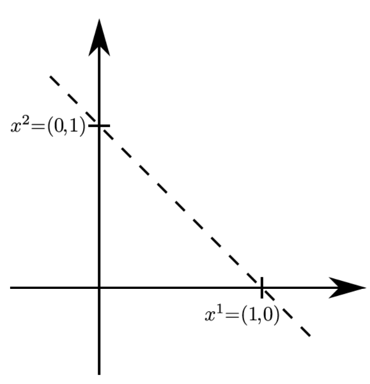

FIGURE 3. aff$(\{x^1, x^2\})$

Assume $s_1 + s_2 = 1$. Then $s_2 = 1 - s_1$. If $x \in$ aff$(\{x^1, x^2\})$, then $x = s_1 x^1 + s_2 x^2$, i.e. $x = (s_1, 1 - s_1)$. But $\langle (s_1, 1 - s_1), (1, 1)\rangle = 1$. It follows that aff$(\{x^1, x^2\}) = H_\alpha$, where $\alpha(x) = \langle x, (1, 1)\rangle - 1$.

Let α_1 and α_2 be two affine forms. Then

$$\alpha_1(x) = \langle x, a_1 \rangle + b_1;$$
$$\alpha_2(x) = \langle x, a_2 \rangle + b_2.$$

Definition 9. H_{α_1} is **parallel** with $H_{\alpha_2} \Longleftrightarrow H_{a_1} = H_{a_2} \Longleftrightarrow a_1 = \pm a_2$.

Remark 10. (a) If H_{α_1} is parallel with $H_{\alpha_2} \Longrightarrow H_{\alpha_1} \cap H_{\alpha_2} = \emptyset$ or $H_{\alpha_1} = H_{\alpha_2}$.
(b) If H_{α_1} is not parallel with $H_{\alpha_2} \Longrightarrow H_{\alpha_1} \cap H_{\alpha_2}$ is an affine subspace and $\dim H_{\alpha_1} \cap H_{\alpha_2} = p - 2$.

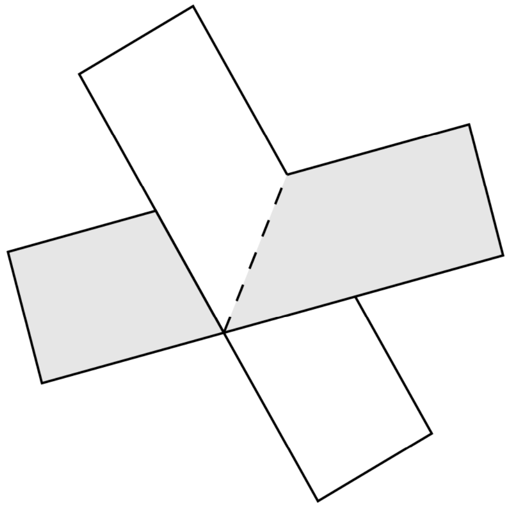

FIGURE 4. Intersection of two hyperplanes

Polytope

Definition 11. A subset $\mathcal{P} \subset \mathbb{R}^p$ is called a **polyhedron** if it is the intersection of finitely many closed halfspaces. Equivalently, $\mathcal{P}$ is a polyhedron if it's the set of solutions of a system of linear inequalities. The **dimension** of $\mathcal{P}$ is $\dim \operatorname{aff}(\mathcal{P})$.

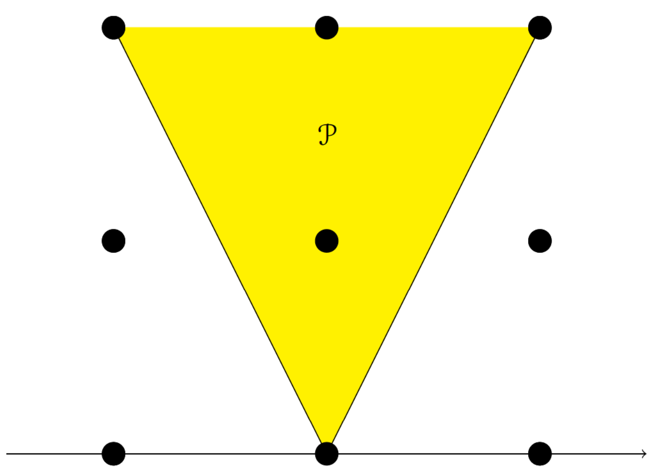

FIGURE 5. A rational polyhedron

Definition 12. A **polytope** is a bounded polyhedron. A 2-dimensional polytope is called a **polygon**.

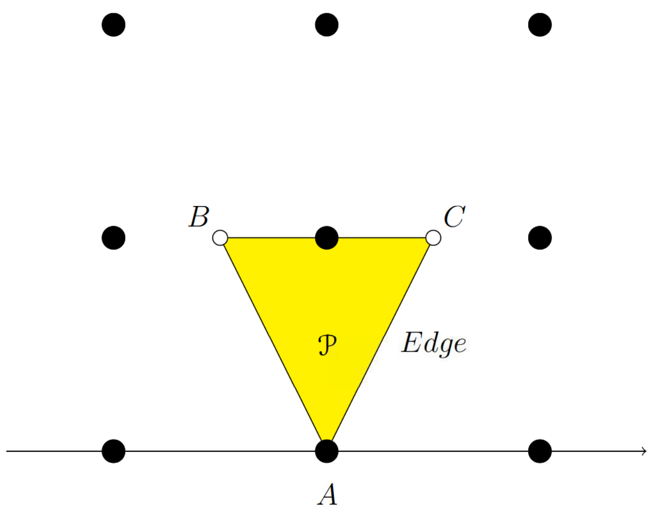

FIGURE 6. A rational polytope

Definition 13. A hyperplane H is called a **support hyperplane** of the polyhedron $\mathcal{P}$ if $\mathcal{P}$ is contained in one of the two halfspaces bounded by H and $H \cap \mathcal{P} \neq \emptyset$. $F = H \cap \mathcal{P}$ is called a **face** of $\mathcal{P}$, and we have that $\dim F = \dim \operatorname{aff}(F)$.

Faces of dimension 0 are called **vertices**.

Faces of dimension 1 are called **edges**.

Faces of dimension $\dim \mathcal{P} - 1$ are called **facets**.

Definition 14. A polytope $\mathcal{P}$ of dim d with precisely $d+1$ vertices is called a **d-simplex**.

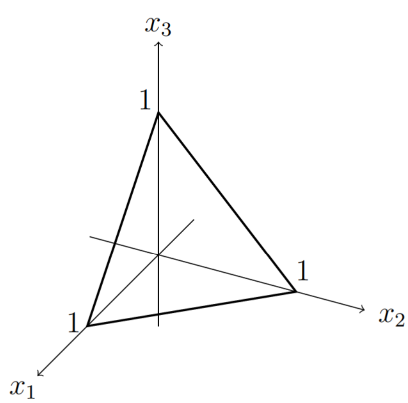

FIGURE 7. The unit simplex in R^3

The **standard simplex** is the polytope with vertices e_j, where e_j is the standard basis in $\mathbb{R}^p \iff$

$\langle x, e \rangle = 1, e = (1, ..., 1);$

$\langle x, e_j \rangle \geq 0.$

Algebraic Equations and Inequalities

(Introduction to Real Algebraic Geometry)

Definition 15. Let $q \in \mathbb{R}[X_1, ..., X_p]$ be a polynomial in p variables. We say that q is **irreductible** if it cannot be factored into the product of two non-constant polynomials.

Definition 16. The set

$$S_q = \{x \in \mathbb{R}^p \mid q(x) = 0\}$$

is called an **algebraic hypersurface**. If q is irreductible, S_q is called **algebraic variety**. If not, it is called an **algebraic set** (that is in fact a union of algebraic varieties).

We can define $S_q^+, S_q^-, S_q^>$ and $S_q^<$ as before.

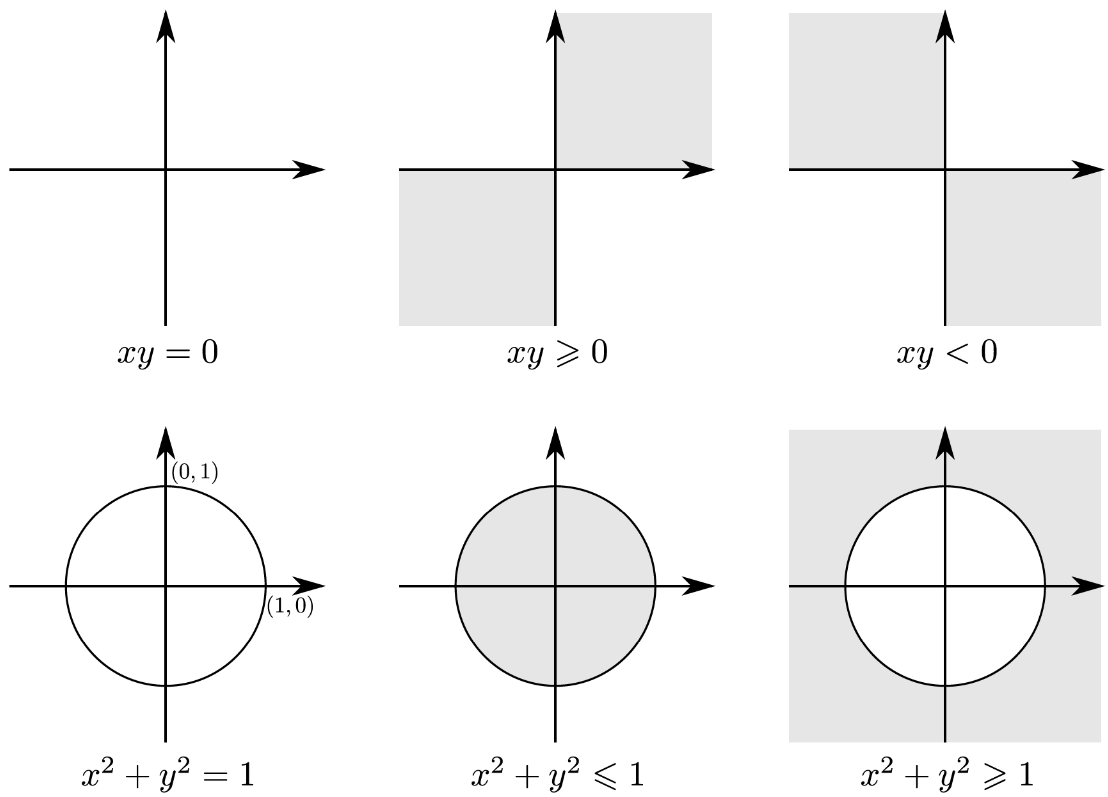

FIGURE 8